Building Intuitive Arguments for the Triangle Congruence Conditions

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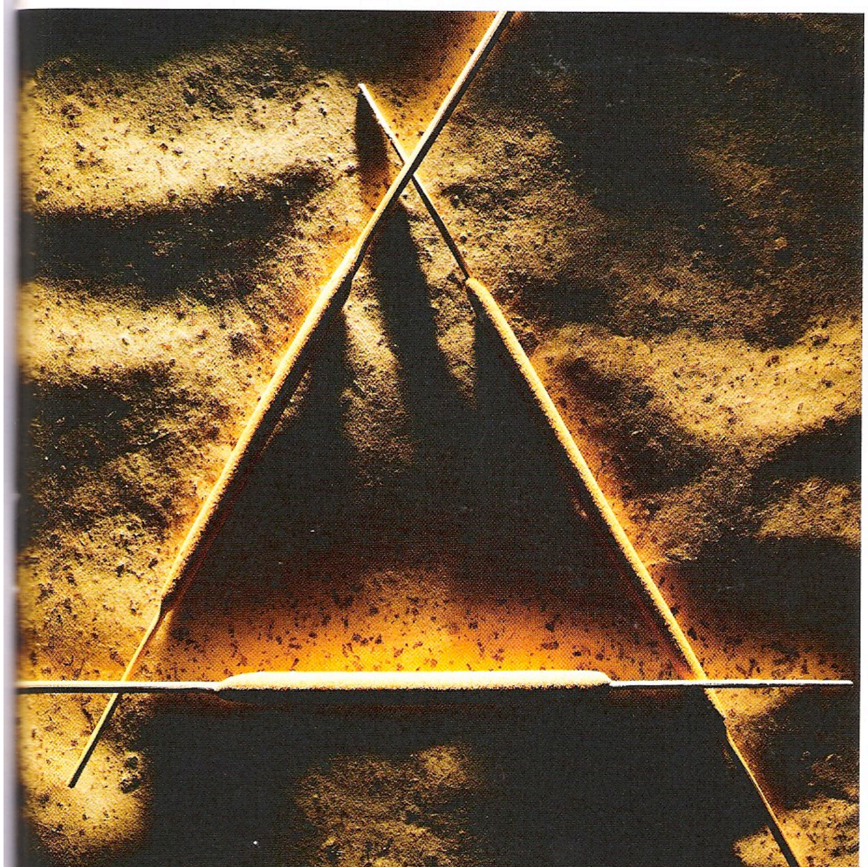

The triangle congruence conditions are a central focus of nearly any course in Euclidean geometry. Although Euclid proved each of these four congruence conditions—SAS, SSS, ASA, AAS—around 300 BCE, the formal proofs of these conditions may be beyond the scope of many high school courses. As a result, in the high school curriculum we often introduce the triangle congruence conditions as postulates rather than theorems.

Geometry textbooks often define *postulates* as mathematical statements accepted without proof. It is natural, however, for students to question several aspects of postulates:

- *Why* do we accept some mathematical statements without proof and not others?
- *Which* mathematical statements may be accepted without proof and which statements must be proven?
- *Why* do we believe these statements are true?

One high school textbook, *Informal Geometry* (Smith et al. 1992) begins to answer the first of these questions by comparing postulates to rules of a game: "In the same way that people playing a game must agree beforehand on the rules, mathematicians have agreed to begin the study of geom-

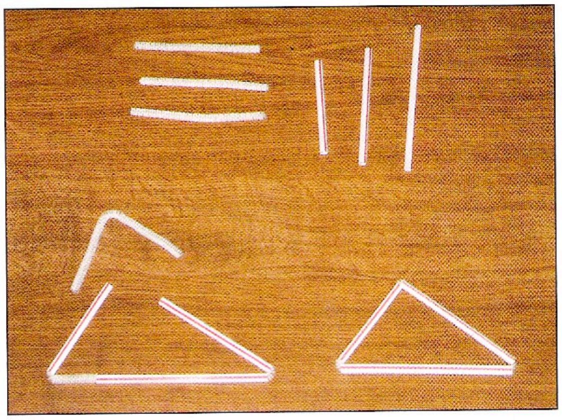

Fig. 1 Materials for constructing triangles

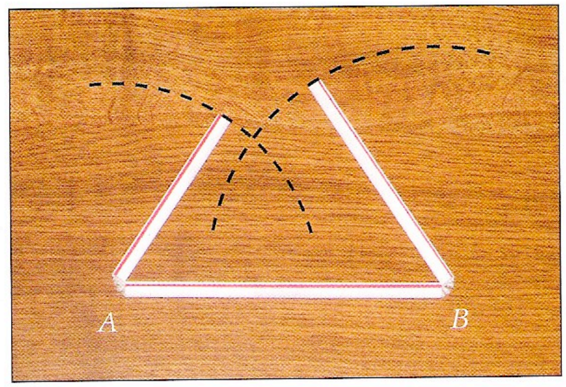

Fig. 2 Only one possible triangle can be created given three side lengths.

etry with certain statements that they assume to be true. The statements that mathematicians assume true are called postulates" (p. 44). Another textbook, *Focus on Geometry: An Integrated Approach* (Hoffer and Koss 1998), makes an analogy between the assumptions made in the mathematical system of geometry and the self-evident truths on which the U.S. government is founded: "The Continental Congress used the assumptions stated in the Declaration of Independence in writing the original United States Constitution in 1787.... Mathematical systems are also built on assumptions" (p. 114).

Although these excerpts provide explanations as to why postulates are necessary in mathematics, they do not address such questions as which statements are chosen to be accepted as postulates or why we believe these statements are true. When students have the opportunity to explore possible triangle congruence conditions, they are able to develop arguments for themselves as to whether or not each possible combination of As and Ss forms actual congruence conditions. Such justification allows the students to accept these statements as true and provides them with a conceptual understanding of the meaning of the statements.

In the following activity, students use straws and pipe cleaners to explore possible triangle congruence conditions. Students could simply draw pictures of triangles, but using three-dimensional objects, such as straws, has many benefits. Have you ever attempted to draw a triangle with sides of length 3 inches, 4 inches, and 6 inches by using only a ruler and a pencil? It is difficult to know how large the angles should be. Straws and pipe cleaners allow for movement and, thus, easier construction of a triangle with these side lengths. Furthermore, the use of straws encourages students to develop dynamic mental images of geometrical objects. The ability to transform mathematical objects mentally as well as anticipate the results of transformations is necessary for students to develop what Harel and Sowder (1998) refer to as a transformational proof scheme, one of the two analytical proof schemes we would like our students to obtain. (The other analytical proof scheme, called the axiomatic proof scheme, shows that a student understands that mathematical systems are necessarily built from postulates and undefined terms. Discussions on the need for postulates and the difference between postulates and theorems are excellent ways to nurture students' development of this proof scheme.) Another benefit of using the straws with pipe cleaners is that a three-dimensional triangle is a better visual aid than a two-dimensional drawing.

DESCRIPTION OF THE ACTIVITY

Before we discuss any of the triangle congruence conditions, I provide each student with three straws, a piece of pipe cleaner, a pair of scissors, and a ruler. I ask each student to cut the pipe cleaner into three equal pieces and then to create a triangle by using the straws for the triangle's sides and the pipe cleaner as connectors (see **fig. 1**). Then I ask each student to create a triangle with sides of length 3 inches, 4 inches, and 6 inches, using the ruler and a scissors to cut the straws. When the students have created their triangles, they compare theirs with those created by the other students in the class. They soon discover that all the triangles are congruent to one another.

At this point, I ask the students to construct arguments as to why all the triangles in the room turned out to be congruent and whether any two triangles with sides of length 3 inches, 4 inches, and 6 inches will necessarily be congruent. One typical response is, "Yes, any two triangles with sides of these lengths will be congruent because no matter which order you place the straws, you will always get the same triangle, just maybe turned around." A response reflecting a more dynamic approach might be something like this: "If you place the 6-inch straw on the table and attach the other two straws at the ends of it, then there is only one way for these three straws to make a triangle, when the ends of the 3-inch straw and the 4-inch straw meet" (see **fig. 2**). Although it is true that these two straws will

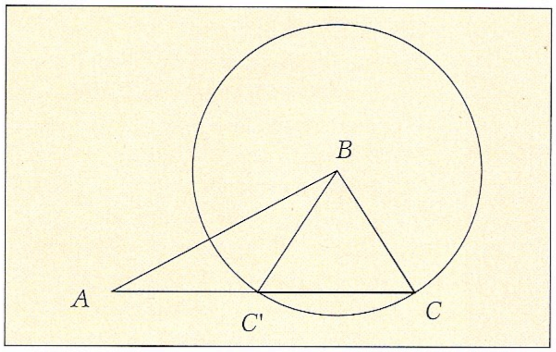

Fig. 3 Given SSA with $\angle A$, $\overline{AB}$, and $\overline{BC}$ such that $AB > BC > AB \cdot \sin A$

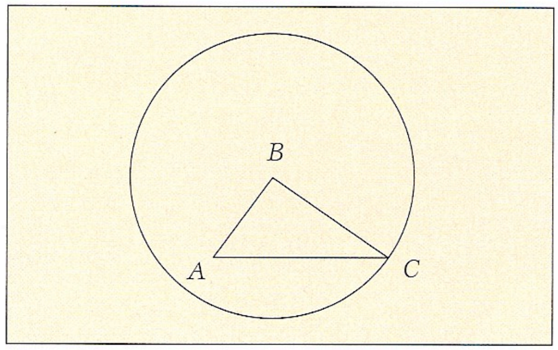

Fig. 4 Given SSA with $\angle A$, $\overline{AB}$, and $\overline{BC}$ such that $AB < BC$

meet once more on the opposite side of the 6-inch straw, with a little exploration the students can see that this "other" triangle is simply a mirror image of the first and, therefore, is congruent to it.

Now that the class as a whole has conjectured that knowing the lengths of the three sides of a triangle is enough to determine whether it is a unique triangle, I have students explore these questions: Will one side be enough to determine a triangle? Two sides? One angle? Two angles? One angle and one side? Because visualizing an angle when we do not know one or both of its composing side lengths is sometimes difficult, I generally have the students use a bent piece of pipe cleaner to represent such an angle. It is important for the students to draw a variety of acute and obtuse angles as well as a right angle in order to ensure that they have considered a variety of cases.

Once the class establishes that at least three parts of a triangle are needed to determine whether it is a unique triangle, we decide as a class how many unique combinations of sides and angles exist. I generally write the six combinations on the board—SSS, SAS, SSA, ASA, AAS, and AAA—and divide the class into five groups, asking each group to explore one of the remaining five possibilities. The students in each group must either produce a counterexample showing why their combination does not work or devise a justification as to why it does. They then present their work to the class in order to convince classmates of their argument.

Note that SSA is the most difficult of these combinations because it does hold as a congruence condition in the special case where the longer given side is opposite the given angle. Consider SSA with $\angle A$, $\overline{AB}$, and $\overline{BC}$ such that $AB > BC > AB \cdot \sin A$, as shown in **figure 3**. (Unless $BC > AB \cdot \sin A$, no triangle will be formed.) Then, $\overline{BC}$ is allowed to rotate freely around point B, since ΔABC is not fixed. This allows for two noncongruent triangles (ΔABC and $\Delta ABC'$) to exist with the given information such that $\Delta ABC \not\equiv \Delta ABC'$. As in **figure 4**, however, consider SSA with $\angle A$, $\overline{AB}$, and $\overline{BC}$ such that $AB < BC$ as shown in **figure 4**. Then, when $\overline{BC}$ rotates freely around point B, it does not intersect the ray $\overline{AC}$ again, showing that only one possible triangle exists that conforms to the given information.

This attribute of SSA presents an excellent opportunity for discussing how mathematical theorems are developed. I explain that when stated analogously to the other congruence conditions, SSA is not a congruence condition. If one were to place more restrictions on the hypothesis, however, one could come up with different mathematical statements involving two sides of a triangle and a nonincluded angle (Hirschhorn 1990). One such restriction frequently mentioned in geometry textbooks is known as the hypotenuse-leg theorem (HL theorem) and occurs when the given angle is a right angle.

BENEFITS OF THE ACTIVITY

Conducting such an activity with a class has many benefits. First, students have a chance to hone both inductive and deductive reasoning skills. Although they are not expected to devise formal proofs for these congruence conditions, it is not uncommon for students to provide contradiction arguments within their explanations. This may be a result of the dynamic nature of this activity, which enables students to envision transformations occurring within geometric objects. Harel and Sowder (1998) discuss the importance of conceptualizing transformations of mathematical objects in developing mathematical justification and formal mathematical proof.

Through this activity, students can explore and investigate mathematical ideas and construct mathematical conjectures, as recommended by the National Council of Teachers of Mathematics (2000). By presenting their thoughts to the class, students practice communicating their mathematical thinking to others. Furthermore, students are able to see that mathematics is not simply a list of rules to memorize—SSS, SAS, ASA, and AAS satisfy triangle congruence conditions while SSA and AAA do not—but that mathematics can be reasoned. Finally, my students find working with straws and pipe cleaners fun!

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